Fracture Toughness Relation in Biaxial Loading

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The fracture toughness relation (FTR) in biaxial loading of a body containing a slanting crack has been analyzed. A basic requirement for the validity of the FTR is discussed. A criterion based on the energy release rate in each mode is presented and compared with the experimental data.

Keywords energy criterion, fracture, mixed mode

1. Introduction

The fracture criterion of bodies with slanting cracks has been investigated by many researchers[1–11] based on the stress distribution at the crack tip and the associated strain energy. Notable among them are (a) the minimum strain energy density criterion,[1,2] (b) the maximum circumferential stress criterion,[4] and (c) the maximum energy release criterion.[5]

The minimum strain energy density criterion[1,2] leads to

$$a_{11}\,(K_I)^2 + a_{12}\,(K_I)\,(K_{II}) + a_{22}\,(K_{II})^2 = S \quad \text{(Eq 1)}$$

where a_{11}, a_{12}, and a_{22} are constants; and S is the critical value.

Ueda *et al.*[6] have given the relation for the strain energy release rate and the associated SIF as

$$G = \frac{1 - \mu^2}{E}\,[(K_I)^2 + (K_{II})^2 + (1/1 - \mu)\,(K_{III})^2] \quad \text{(Eq 2)}$$

In all the strain-energy-based approaches, a core region just ahead of the crack is considered and the strain energy in that region is computed and equated to a critical value. The shape and size of the core region change depending on the assumptions made.

Vishu[12] has carried out an experimental study of the mixed mode crack propagation and has suggested a relation of the type

$$0.05\,(K_I)^2 + 0.95\,(K_I) + 2.16\,(K_{II})^2 + 1.98\,(K_{III})^2 = 1 \quad \text{(Eq 3)}$$

where $[K]$ indicates the normalized SIF (the SIF divided by K_{Ic}).

These theories relate the stress intensity factors (SIFs) in modes I and II acting on the slanting crack at the fracture separating the "Fracture" and "No Fracture" zones on the K_I-K_{II} plane. The merits and deficiencies of the different theories are discussed in Ref. 13.

A point (K_I, K_{II}) on the K_I-K_{II} plane gives either a Fracture or a No Fracture condition only, which is decided by the values of K_I and K_{II} acting on the slanting crack. In the above analyses, the SIF due to compressive stress on the crack and the T stress, *i.e.*, the normal stress acting parallel to the crack plane, are assumed to play insignificant roles in the fracture process.

In the following, an analysis for the basic requirement of mixed mode fracture in biaxial loading is presented based on the Mohr's type of SIF semicircles corresponding to the uniaxial and pure shear type of loading in the range of the biaxiality ratio λ (normal SIF in the principal direction 2/normal SIF in the principal direction 1 = K_{I2}/K_{I1}) from zero (uniaxial condition) to −1 (pure shear condition). A fracture toughness relation (FTR) based on the sum of the energy release rates in each mode

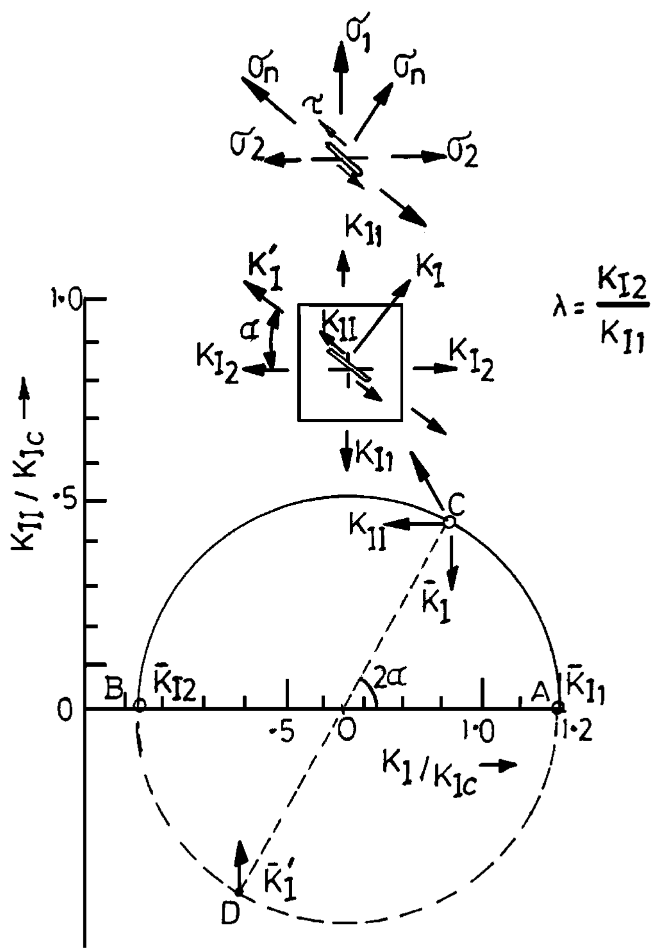

Fig. 1 Stress Intensity Factor circle

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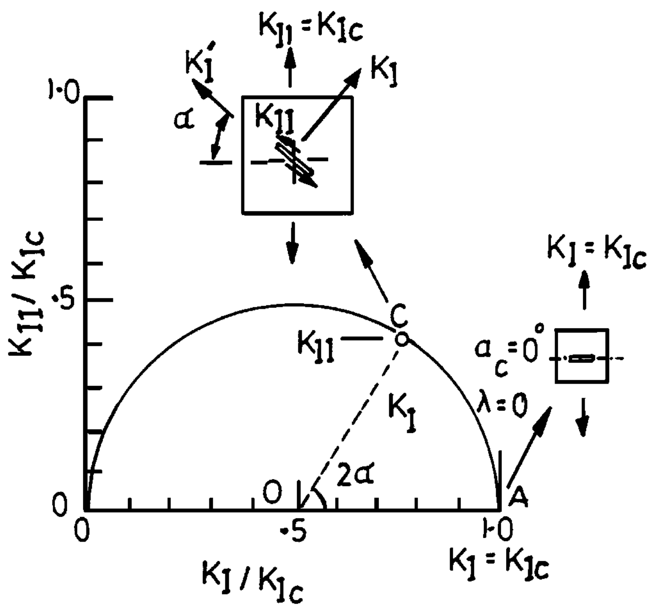

Fig. 2 SIF Semi-Circle for uni-axial loading condition with $K_{I1} = K_{Ic}$

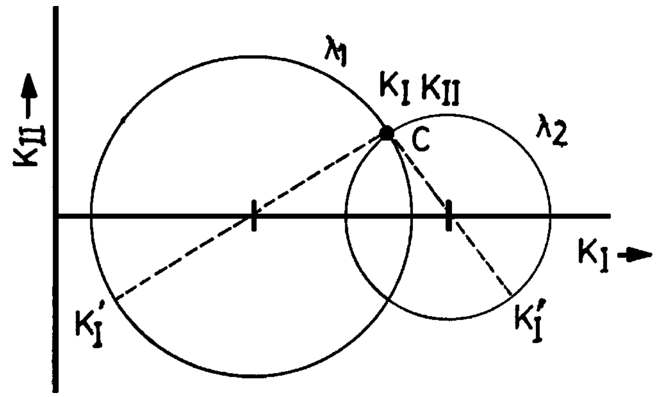

Fig. 3 Different loading systems producing the K_I and K_{II}

is considered and the relation obtained is compared with the experimental data.

2. SIF Semicircle

Figure 1 shows a through crack in a large body, which is subjected to normal stresses σ_1 and σ_2 in the principal directions 1 and 2. When the crack is inclined at an angle α to the direction 2, the crack experiences a normal stress, σ_n, perpendicular to its plane, $\sigma n'$, parallel to the plane and the shear stress τ. The normal stresses σ_n and σ_n' will give rise to SIFs K_I and K_I' of which K_I' (notional value) will not affect the crack as it is parallel to the crack plane. The shear stress τ will give rise to SIF K_{II}, as indicated in the figure. The crack plane is taken to be parallel to the third direction and $K_{III} = 0$.

The figure also shows the SIF circle, similar to the Mohr's circle, for the biaxial condition. K_{I1} and K_{I2} are the SIFs corresponding to the principal stresses σ_1 and σ_2 when the value of the crack angle a is equal to 0 and 90°, respectively. These conditions are indicated by the points A and B on the X-axis. The bar indicates normalized SIF through dividing by K_{Ic}. When the crack is inclined at an angle a, the crack plane is given by the line CD and the values of K_I and K_{II} are given by the point C. The point D gives the notional SIF K_I'. It is assumed that the T stress acting parallel to the crack plane does not influence the fracture process and so the notional SIF K_I' is not considered for further discussion and, hence, the lower portion of the SIF circle is shown by dotted lines. Only the upper half of the circle gives the required K_I and K_{II} acting on the inclined crack.[14]

3. SIF Semicircle and Fracture in Uniaxial Loading

Consider a material under uniaxial loading and whose fracture toughness ratio K_{IIc}/K_{Ic} ($=m$) is such that $0.5 < m < 1.0$. When the principal stress σ_1 is such with the critical crack angle $\alpha_c = 0$, the stress value gives the critical SIF $K_{I1} = K_{Ic}$ and so fracture occurs. Under this uniaxial condition, as shown in Fig. 2, when the crack becomes inclined with a different crack angle α, the normal SIF K_I and the SIF due to shear stress K_{II} acting on the crack are given by the SIF semicircle. Fracture will occur, however, only when $\alpha = \alpha_c =$ zero. For other inclined cracks, fracture will not occur because the SIF $K_{I1} = K_{Ic}$ corresponds to the crack angle $\alpha_c = 0$. For any other inclined crack, K_{I1} has to be further increased (*i.e.*, σ_1 has to be further increased) beyond K_{Ic} to cause fracture. Thus, the points (such as point C) on the SIF semicircle with $\lambda = 0$ and $K_{I1} = K_{Ic}$ represent the K_I and K_{II} relation for the No Fracture condition.

Now let us consider a point "C" (K_I, K_{II}) on the K_I versus K_{II} plane, as shown in Fig. 3. This point C (K_I, K_{II}) can be produced by different loading systems such as $\lambda 1$ and $\lambda 2$ represented by the respective SIF circles. But the point C (K_I, K_{II}) will represent either a Fracture condition or a No Fracture condition only, irrespective of the loading system λ (K_{I2}/K_{I1}), which produces these SIFs K_I and K_{II}, as indicated in the figure. If a point C can represent both conditions, then there will not be any unique FTR between K_I and K_{II}. Thus, the SIF semicircle obtained based on the uniaxial loading condition ($\lambda = 0$ and $K_{I1} = K_{Ic}$), as shown in Fig. 2, will represent a No Fracture relation between K_I and K_{II} even if the loading system is altered, *i.e.*, for different sets of K_{I1}, K_{I2} and crack angle α conditions.

4. SIF Semicircle and the FTR

Now the SIF semicircle discussed above and the FTR in biaxial loading (FTR) can be clubbed together to consider the Fracture and No Fracture conditions. Figure 4(a) shows a hypothetical FTR relation and the SIF semicircle for uniaxial loading. The FTR separates the Fracture and No Fracture zones on theK_I-K_{II} plane. In the present example, the FTR intersects the SIF semicircle at C, as seen in the figure. The points on the right side of the SIF semicircle lie in the fracture zone, as predicted by the criterion assumed in the development of the FTR. However, these points lying on the semicircle indicate a No Fracture condition, as discussed in the previous section. Since any point on the K_I-K_{II} plane can represent either a No Fracture condition or a Fracture condition only, and since it is shown that the points on the SIF semicircle ($\lambda = 0$, $K_{I1} = K_{Ic}$) represent a No Fracture condition, it is clear that the FTR should not intersect the SIF semicircle ($\lambda = 0$, $K_{I1} = K_{Ic}$), but it should be tangential, and just touch the SIF semicircle at $K_I = K_{Ic}$ on the *X*-axis. Thus, the assumptions made in the derivation of the FTR between K_I and K_{II} require reconsideration.

Similar to the SIF semicircle, which gives the No Fracture relation between K_I and K_{II} obtained by considering uniaxial loading with $\lambda = 0$ and $K_{I1} = K_{Ic}$, the SIF semicircle corresponding to pure shear loading (*i.e.*, $\lambda = -1$ and $K_{I1} = -K_{I2} = K_{IIc}$) will also give the relation between K_I and K_{II} for the No Fracture condition except when the crack angle $\alpha = 45°$. Figure 4(b) shows a portion of the SIF semicircle for $\lambda = -1$, $K_{II} = K_{IIc}$, along with the hypothetical FTR. The FTR intersects the SIF semicircle at C, and the points left of C on the semicircle are lying in the fracture zone predicted by the FTR. However, the points on the SIF semicircle represent a No Fracture condition except at $K_{I1} = K_{IIc}$ and $\alpha = \alpha_c = 45°$. Thus, the FTR should not intersect the SIF semicircle for $\lambda = -1$ and $K_{I1} = -K_{I2} = K_{IIc}$, but it should just touch and be tangential to the semicircle at $K_{II} = K_{IIc}$ on the *Y*-axis.

Since these SIF semicircles form the basis for validation of any theoretical FTR, they are termed validity semicircles. These two validity SIF semicircles ($\lambda = 0$, $K_{I1} = K_{Ic}$) and ($\lambda = -1$, $K_{I1} = -K_{I2} = K_{IIc}$) for positive values of K_I are shown in Fig. 5, and any point (K_I, K_{II}) on or inside these semicircles indicates a No Fracture condition.

5. Lower Bound of FTR

It is clear from the foregoing analysis that the FTR between K_I and K_{II} based on any criterion should not intersect the two validity SIF semicircles ($\lambda = 0$, $K_{I1} = K_{Ic}$) and ($\lambda = -1$, $K_{I1} = -K_{I2} = K_{IIc}$). The FTR relating K_I and K_{II} should be tangential and just touch the validity semicircles at $K_{II} = K_{IIc}$ on the *Y*-axis and at $K_I = K_{Ic}$ on the *X*-axis. This validity requirement must be satisfied by any theoretical FTR. Thus, it can be seen that these SIF semicircles form the lower bound of the K_I and K_{II} relation for fracture. Figure 6 shows the lower bound of FTR for the two conditions, namely, $0.5 < m < 1.0$. The experimental data taken from Ref 10 are lying above these validity SIF semicircles.

Some materials such as plastics may be weak in shear with the result that K_{IIc} may be lower than 0.5 K_{Ic}, *i.e.*, the value of $m < 0.5$. In brittle materials including concrete, the K_{IIc} value could be higher than the K_{Ic} value, resulting in $m > 1.0$. The lower bound diagram for *a* range of *m* values is shown in Fig. 7.[15] When K_{IIc} is less than 0.5 K_{Ic}, we get ABC, the lower bound line. In uniaxial loading above a certain angle α_1 of the crack, the K_{II} component acting on the crack will be greater than K_{IIc} and so fracture is likely to occur with crack angles greater than α_1 in the uniaxial loading.

When $m = 1$, the lower bound is FC and is part of the semicircle $\lambda = -1$. When *m* is greater than one, it can be seen that, when the validity circle $\lambda = -1$ is drawn, it will cut the vertical line CH through the point C ($K_I = K_{Ic}$). This means that in pure shear for cracks with crack angles less than α_2, the normal SIF component K_I will be greater than K_{Ic}, and so for cracks with angles from α_2 to zero, fracture may occur in pure shear loading also. The experimental data points are taken from Ref 10.

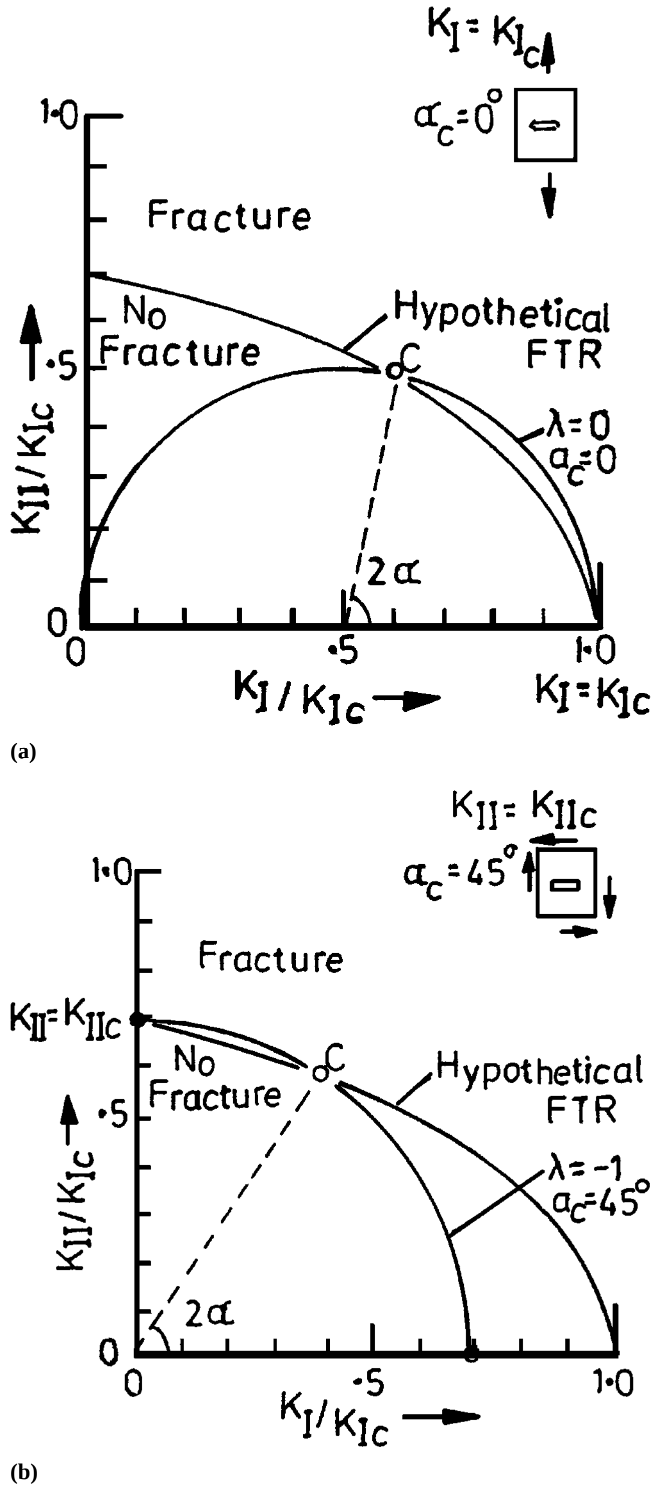

Fig. 4 **(a)** SIF semi-circle ($\lambda = 0$, $K_{I1} = K_{Ic}$) and the hypothetical FTR; **(b)** SIF semi-circle ($\lambda = -1$, $K_{I1} = -K_{I2} = K_{IIC}$) and the hypothetical FTR

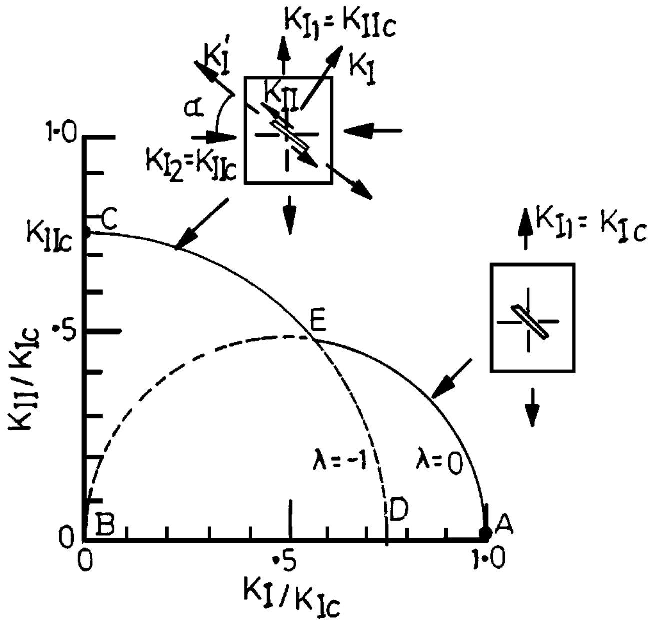

Fig. 5 The two validity SIF semi-circles ($\lambda = 0$, $K_{I1} = K_{Ic}$) and ($\lambda = -1$, $K_{I1} = -K_{I2} = K_{IIc}$)

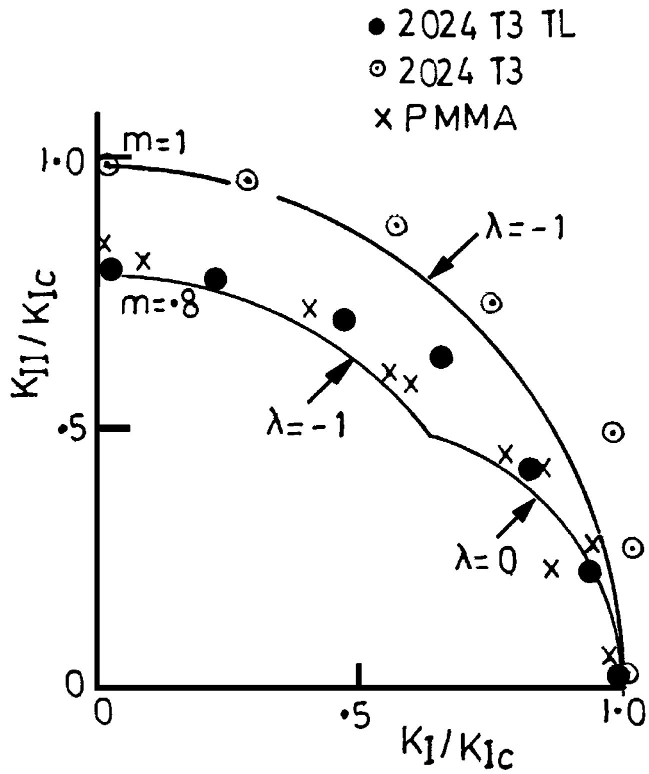

Fig. 6 Lower limits of toughness relation in bi-axial loading and data correlation

6. Theoretical Consideration Based on Energy Release Rate

In the present approach, it is taken that, when the energy release rate in each mode reaches the critical value in the particular mode, fracture will occur. This leads to

$$G_I/G_{Ic} = 1;\ G_{II}/G_{IIc} = 1;\ \text{and}\ G_{III}/G_{IIIc} = 1 \quad \text{(Eq 4)}$$

Extending this condition that the maximum energy release rate G in each mode is the controlling factor even in mixed mode condition, and that fracture occurs when the sum of the ratios of the energy release rates to their respective critical values in each mode is equal to one, this is

$$\Sigma(Gi/Gic) = 1 \quad \text{(Eq 5)}$$

where i = I, II, or III. The interactive relation between K_I, K_{II}, and K_{III} can be written for multiaxial loading as

$$(K_I/K_{Ic})^2 + (K_{II}/m\ K_{Ic})^2 + (1/1 - \mu)\ (K_{III}/m'\ K_{Ic})^2 = 1 \quad \text{(Eq 6)}$$

where m and m' are given by

$$K_{IIc} = m\ K_{Ic} \text{ and } K_{IIIc} = m'\ K_{Ic} \quad \text{(Eq 7)}$$

The values of the constants m and m' are obtained experimentally. This procedure will enable us to take care of the directional property of the material, which, otherwise, may not be possible in other types of criteria. For high strength metallic materials, the value of m may vary from 0.7 to 1.0.

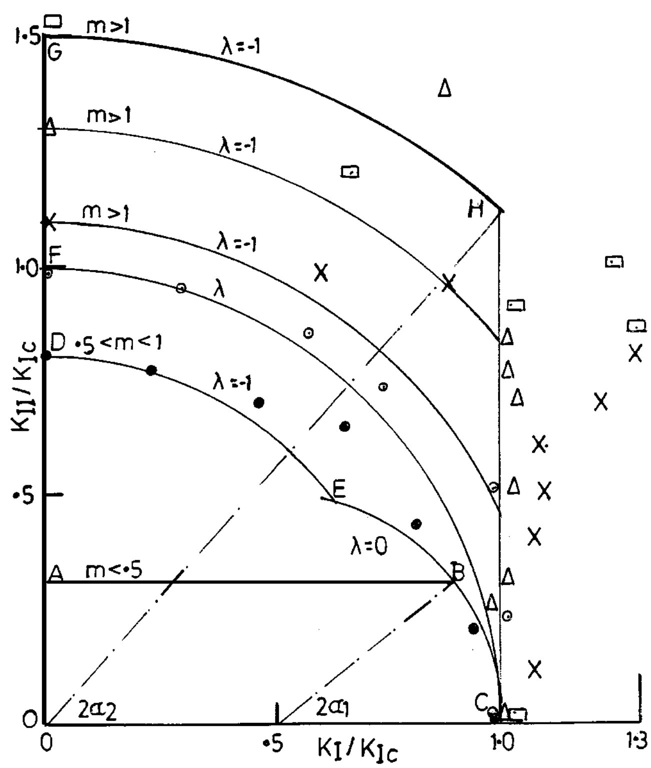

Fig. 7 Lower limits of toughness relation in bi-axial loading for different "m" values

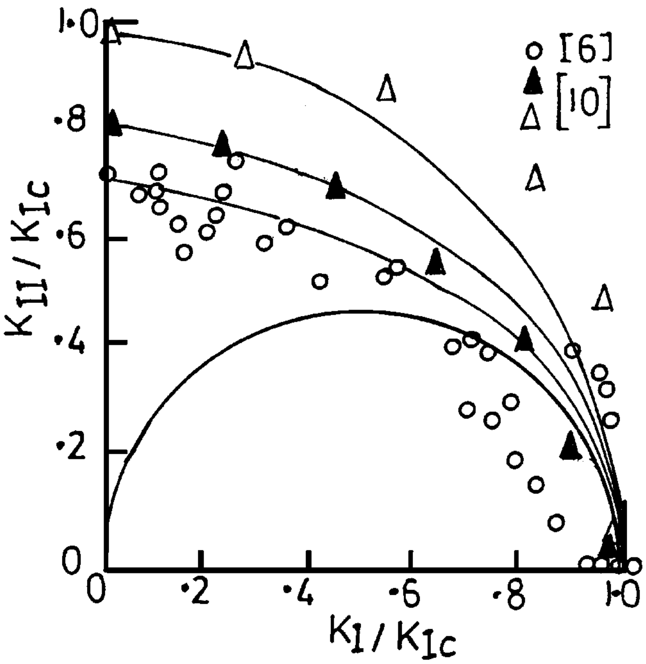

Fig. 8 Relation between K_{II} and K_I

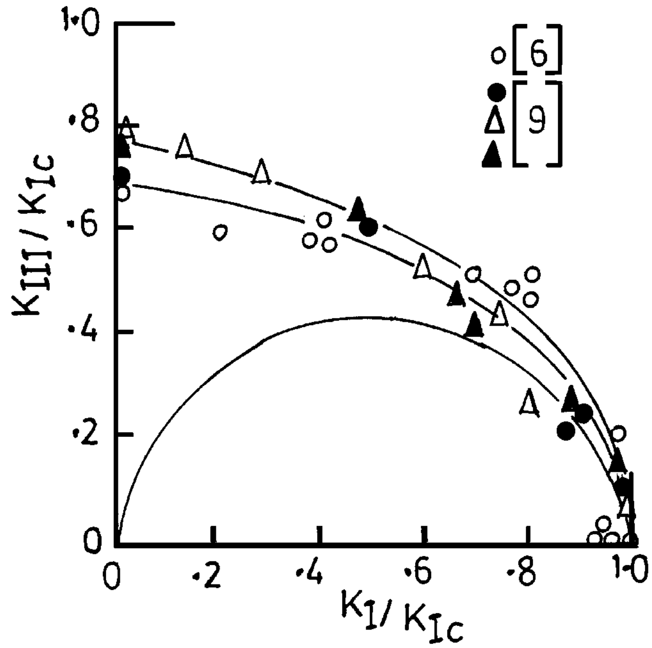

Fig. 9 Relation between K_{III} and K_I

In the case of elastic plastic fracture, Eq 5 can be written as

$$J_I/J_{Ic} + J_{II}/J_{IIc} = 1 \qquad \text{(Eq 8)}$$

which yields a straight line on the plane J_I versus J_{II} at fracture, as has been observed by Sha Jiangbo *et al.*[16]

7. Correlation with Experimental Data

Figure 8 shows the relation between K_I and K_{II} for PMMA and 2024 Ti Al alloy, crack being perpendicular and parallel to the rolling direction. The material PMMA, which is brittle, shows large scatter, the data points being from Ref 6 and 10. The values of m are 0.7, 0.8, and 0.97 for the results shown, and the governing equation is

$$(K_I)^2 + (K_{II}/m)^2 = 1 \qquad \text{(Eq 9)}$$

The validity semicircles in uniaxial and pure shear loadings are given by

$$(K_I - 0.5)^2 + (K_{II})^2 = (0.5)^2 \qquad \text{(Eq 10)}$$

and

$$(K_I)^2 + (K_{II})^2 = (m)^2 \qquad \text{(Eq 11)}$$

The predicted curves do not intersect the validity circle ($\lambda = 0$, $K_{I1} = K_{Ic}$) and they have a common tangent at $K_I = K_{Ic}$ on the x-axis and $K_{II} = K_{IIc}$ on the Y-axis. Thus, the essential condition is also fulfilled by the criterion assumed in this analysis.

Figure 9 shows the relation between K_I and K_{III} with the governing equation

$$(K_I)^2 + (1/1 - \mu)\,(K_{III}/m')^2 = 1 \qquad \text{(Eq 12)}$$

The value of m′ $\sqrt{(1 - \mu)}$ is obtained from the experimental data as 0.67 and 0.75 for the two predictive curves shown (data from Ref 6 and 9). The validity semicircle in this case is given by

$$(K_I - 0.5)^2 + 1/(1 - \mu)\,(K_{III})^2 = (0.5)^2 \qquad \text{(Eq 13)}$$

Due to the factor $1/(1 - \mu)$, the validity circle presents an elliptical shape in the K_I and K_{III} plane. At $K_I = 0.5$, the value of K_{III} is not 0.5, but it is 0.5 $\sqrt{(1 - \mu)} = 0.42$. The predicted fracture toughness lines do not intersect the validity ellipse in the present case.

For the mixed mode condition with the three components acting on the cracked body, the general relation is given in the form

$$(K_I)^2 + (K_{II}/m)^2 + 1/(1 - \mu)\,(K_{III}/m')^2 = 1 \qquad \text{(Eq 14)}$$

Figure 10 shows the relation between K_{III} and K_{II} with $m = 0.7$ and $m' \sqrt{(1 - \mu)} = 0.67$ for different values of K_I. The FTR, as obtained by the above equations, is seen to predict well the experimental data from Zhao.[7]

8. Concluding Remarks

From the study carried out to investigate fracture toughness under mixed mode condition, the following conclusions are derived.

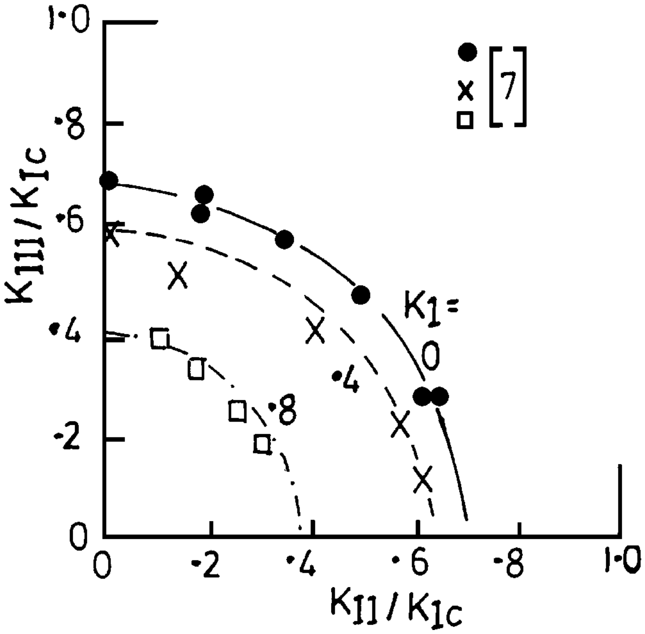

Fig. 10 Relation between K_{III} and K_{II}

- The SIFs K_I and K_{II} acting on a slanting crack can be obtained by the SIF semicircle similar to the Mohr's circle.
- The SIF semicircle under uniaxial loading condition with $\lambda = 0$, $K_{I1} = K_{Ic}$ and the crack angle $\alpha_c = 0$ gives the relation between K_I and K_{II} on any slating crack, which will not cause fracture of the body even if the loading system is changed.
- The SIF semicircle for pure shear loading condition with $\lambda = -1$, $K_{I1} = -K_{I2} = K_{IIc}$ and the crack angle $\alpha_c = 45°$ gives the relation between K_I and K_{II} on any slanting crack, which will not cause fracture even if the loading system is changed.
- The FTR between K_I and K_{II} based on any criterion should not intersect the above two SIF semicircles, but be tangential and touch these semicircles at $K_I = K_{Ic}$ on the X-axis and $K_{II} = K_{IIc}$ on the Y-axis.
- Assuming that the critical energy release rates G_{Ic}, G_{IIc}, and G_{IIIc} in each mode control the fracture in the multiaxial loading, and when the sum of the ratios of the energy release rates to their respective critical values in each mode attains unity fracture occurs, a relation for the fracture of the body containing a slanting crack is proposed, which appears to describe well the experimental data.

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